

Distribution of relative velocities in turbulent aerosols

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We compute the distribution of relative velocities for a one-dimensional model of heavy particles suspended in a turbulent flow, quantifying the caustic contribution to the moments of relative velocities. The same principles determine the corresponding caustic contribution in d spatial dimensions. The distribution of relative velocities $\Delta \mathbf{v}$ at small separations R acquires the universal form $\rho(\Delta \mathbf{v}, R) \sim R^{d-1} |\Delta \mathbf{v}|^{D_2-2d}$ for large (but not too large) values of $|\Delta \mathbf{v}|$. Here D_2 is the phase-space correlation dimension. Our conclusions are in excellent agreement with numerical simulations of particles suspended in a randomly mixing flow in two dimensions, and in quantitative agreement with published data on direct numerical simulations of particles in turbulent flows.

PACS numbers: 05.40.-a, 92.60.Mt, 45.50.Tn, 05.60.Cd

Collisions of particles in randomly mixing or turbulent flows ('turbulent aerosols') have been studied intensively for several decades. The main goal, not yet achieved, is to find a reliable parameterisation of the collision kernel. This is an important question since the frequency of collisions between suspended particles determines the stability of turbulent aerosols. Direct numerical simulations of particles in turbulent flows (see for example [1, 2]) show that collision velocities (and thus the collision kernel) increase precipitously as the 'Stokes number' St is varied beyond a threshold. This dimensionless parameter, $St = (\gamma\tau)^{-1}$, is defined in terms of the particle damping rate γ and the relevant correlation time τ of the flow - both explained below. In [3] (see also [4]) this steep increase was attributed to the fact that singularities (so-called 'caustics') in the particle dynamics result in large relative velocities at small separations. Caustics occur when phase-space manifolds describing the dependence of particle velocity upon particle position fold over. In the fold region, the velocity field at a given point in space becomes multi-valued, giving rise to finite velocity differences between particles at the same position. In [3] it was argued that the collision rate exhibits an activated St -behaviour, $\exp(-A/St)$, reflecting the frequency at which caustics occur [3, 5-7].

The collision kernel is determined by the distribution of relative velocities of the suspended particles at small spatial separations. In order to calculate this collision kernel from the microscopic equations of motion, it is necessary to understand how caustics determine the distribution of collision velocities, and to compute how caustics affect the moments of this distribution. This is an important question for many reasons, not only because the St -dependence of the collision rate is of fundamental importance in aerosol physics. First, such an approach could show which aspects of the collision dynamics in turbulent aerosols are universal, and which aspects depend on the particular nature of the turbulent velocity fluctuations. Under which circumstances are highly simplified models appropriate (such as a white-noise velocity field with a single spatial scale)? Second, it is known that turbulent aerosol particles form fractal clusters (see [6] and references cited therein). These fractal objects fold

in phase space when the above-mentioned singularities occur. This raises the question: how do fractal spatial fractal clustering and caustic singularities together determine relative velocities in turbulent aerosols?

In this paper we compute the joint steady-state distribution of relative velocities and particle separations, first in a one-dimensional model [8] for relative velocities of inertial particles using perturbation theory. Our results show that the distribution of relative velocities at small separations is a power law, and that the power is determined by the correlation dimension of the steady-state phase-space fractal. We show that this power law is a consequence of caustics, and that caustics make a substantial contribution to relative velocities at small separations, consistent with the picture outlined above, and the findings of Ref. [3]. Relative velocities in higher spatial dimensions are determined by the same principles, and we show that the corresponding power law (Eq. (7) below) is a universal property of turbulent aerosols. It determines, in general, how caustics and fractal clustering give rise to fractal phase-space distributions which in turn determine the collision kernel. Last but not least, our findings explain the results of a recent scaling analysis of the particle-velocity structure functions obtained by direct numerical simulations of turbulent flows [9].

We first analyse the one-dimensional model [8, 10]

$$\ddot{x} = \gamma(u(x, t) - v). \quad (1)$$

Dots denote time derivatives, $v = \dot{x}$ is the particle velocity, γ the rate at which the inertial motion is damped relative to the fluid, and $u(x, t)$ is a Gaussian random field describing the velocity of the fluid. Fig. 1 shows particle trajectories according to (1) and illustrates how a smooth manifold of initial conditions develops fold singularities (caustics). We assume that $u(x, t)$ is characterised by spatial and temporal correlation scales η and τ , and by its typical size, u_0 (the statistical properties of $u(x, t)$ are described in the caption to Fig. 1). The dynamics is then determined by two dimensionless parameters, by the Stokes number St and by the 'Kubo number' $Ku = u_0\tau/\eta$ which characterises the fluctuations of $u(x, t)$ [6, 11]. In turbulent flows the Kubo number is of order unity, but in the following we analyse the

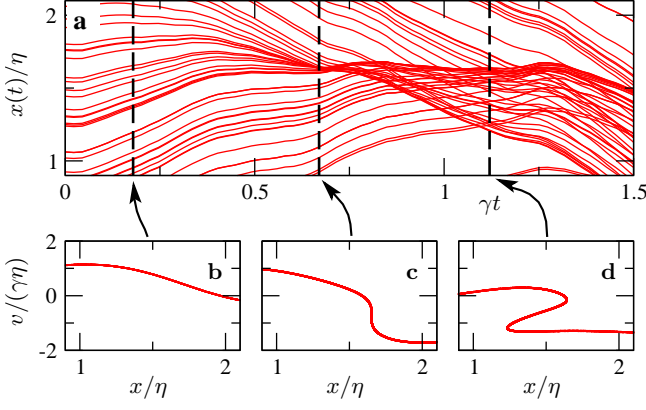


FIG. 1: **a** Particle trajectories $x(t)$ according to Eq. (1) with a Gaussian random field $u(x, t)$, with $\langle u \rangle = 0$ and $\langle u(x, t)u(0, 0) \rangle = u_0^2 \exp[-x^2/(2\eta^2) - |t|/\tau]$ for small values of x , and periodic boundary conditions in x with period L . Parameters: $\tau = 0.01$, $\eta = 0.1$, $L = 1$, $u_0 = 1$, $\gamma = 4/9$, that is $Ku = 0.1$, $St = 225$ (particles initially at rest). Panels **b** to **d** show how the corresponding phase-space manifold folds.

dynamics in the limit of small values of Ku where the suspended particles experience $u(x, t)$ as a white-noise signal. In the limit $Ku \rightarrow 0$, the joint density $\rho(\Delta v, \Delta x)$ is determined by a Kramers equation [11, 12] with a Δx -dependent diffusion constant. In dimensionless units ($t' = \gamma t$, $x' = x/\eta$, $v' = v/(\eta\gamma)$, $u' = u/(\eta\gamma)$) and dropping the primes to simplify the notation) we have:

$$\begin{aligned} \partial_t \rho &= -\partial_{\Delta x}(\Delta v \rho) + \partial_{\Delta v}(\Delta v \rho) + \mathcal{D}(\Delta x) \partial_{\Delta v}^2 \rho, \\ \mathcal{D}(\Delta x) &= \frac{1}{2} \int_{-\infty}^{\infty} dt \langle \Delta u(\Delta x, t) \Delta u(\Delta x, 0) \rangle. \end{aligned} \quad (2)$$

Here $\Delta u(\Delta x, t) = u(x + \Delta x, t) - u(x, t)$. For the model described in Fig. 1 we have $\mathcal{D}(\Delta x) \sim \epsilon^2 \Delta x^2$ for $|\Delta x| \ll 1$ and $\mathcal{D}(\Delta x) \sim 2\epsilon^2 \equiv \mathcal{D}_0$ for $|\Delta x| \gg 1$. Here $\epsilon^2 = Ku^2 St$.

In the limit of $\Delta x \rightarrow 0$, the steady-state solution of (2) is found by separation of the variables Δx and $z = \Delta v/\Delta x$. Inserting the ansatz $g_\mu(\Delta x)Z_\mu(z)$ with separation constant μ into (2) results in $g_\mu(\Delta x) = |\Delta x|^{\mu-1}$, while $Z_\mu(z)$ solves $0 = -\mu z Z_\mu(z) + \partial_z(z + z^2 + \epsilon^2 \partial_z) Z_\mu(z)$.

The steady-state solution of (2) is given by a weighted sum of $g_\mu Z_\mu$ over the allowed values of μ . We know that the distribution of spatial separations exhibits a power law as $\Delta x \rightarrow 0$, corresponding to spatial clustering. We expect that the dominant contribution to this distribution derives from the smallest positive allowed value of μ . For certain values of μ , the functions $Z_\mu(z)$ are known in closed form (for $\mu = 0$ [8] and for $\mu = -1$ [13]). For other values of μ , we determine an analytical solution by expanding Z_μ around $\mu = 0$. We find:

$$\begin{aligned} Z_\mu(z) &= \sum_{k=0}^{\infty} \left(\frac{\mu}{\epsilon^2}\right)^k \int_{-\infty}^0 dt_1 \cdots dt_{2k+1} \left(\prod_{i=1}^k \sum_{j=0}^{2i} t_j \right) \\ &\times \exp\left(-\sum_{i=0}^{2k+1} (-1)^i V\left(\sum_{j=0}^i t_j\right)\right) \end{aligned} \quad (3)$$

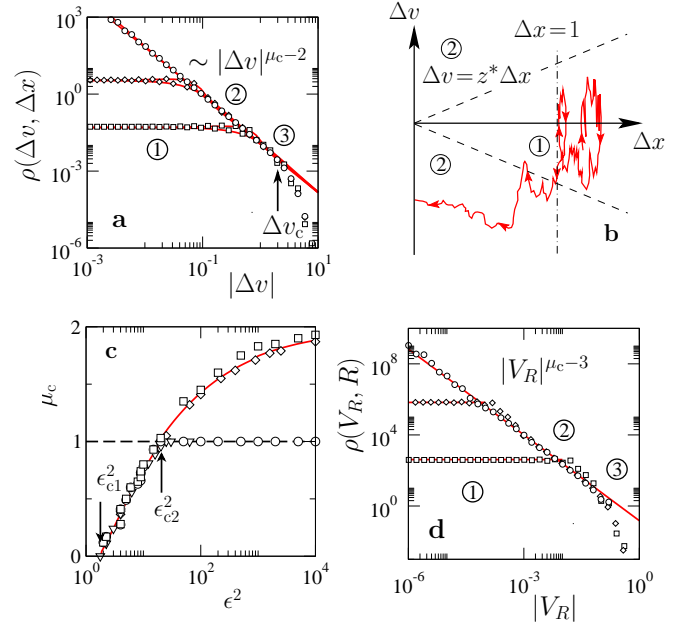


FIG. 2: **a** Steady-state relative-velocity distribution $\rho(\Delta v, \Delta x)$ at $\Delta x = 6 \times 10^{-4}$ (\circ), 5×10^{-2} (\diamond), and 5×10^{-1} (\square), otherwise same parameters as in Fig. 1. Also shown are results from (3) subject to the constraint $A_\mu^+ = A_\mu^-$ (solid red lines). **b** Typical trajectory in the Δx - Δv -plane. Arrows indicate the direction of motion. **c** Results for $\mu_c(\epsilon^2)$ from (3), red line, numerical results for the spatial correlation dimension d_2 (\circ), and numerical results for the phase-space correlation dimension D_2 (\square), both for $Ku = 0.1$. Also shown are results of a linearised Langevin model ($Ku = 0$) for d_2 (∇) and D_2 (\diamond). Two critical values of ϵ^2 are shown: for the path-coalescence transition [8], $\epsilon_{c1}^2 \approx 1.77$, and the second critical value $\epsilon_{c2}^2 \approx 20.7$ where d_2 equals unity. **d** Numerical results (symbols) for the distribution $\rho(V_R, R)$ of relative velocities from computer simulations of a linearised Langevin model for inertial particles suspended in a two-dimensional incompressible Gaussian random flow, $\epsilon^2 = 0.04$, at $R = 5 \times 10^{-6}$ (\circ), 5×10^{-4} (\diamond), and 5×10^{-2} (\square). Also shown is the theoretical result Eq. (8) (solid red line), the value of $\mu_c = D_2$ was determined by numerically computing the phase-space correlation dimension D_2 .

with $V(y) = \epsilon^{-2}(y^2/2 + y^3/3)$ and $t_0 = z$. A related expansion was used in [14] to calculate moments of the finite-time Lyapunov exponent for particles accelerated in a random time-dependent potential. To determine the allowed values of μ we consider the large- z asymptotes of $Z_\mu(z)$. The known solution $Z_0(z)$ [8] exhibits power-law tails $\sim |z|^{-2}$ reflecting the fact that caustic singularities are reached in finite time. Correspondingly, when z is large we can neglect the term $\partial_z(z Z_\mu)$ in the equation for Z_μ . The resulting equation is solved by combinations of Kummer functions with the asymptotic behaviour $Z_\mu \sim |z|^{\mu-2}$ for large values of $|z|$. A power-law ansatz in the equation for Z_μ gives

$$Z_\mu(z) \sim A_\mu^\pm (\pm z)^{\mu-2} (1 + (\mu-1)z^{-1} + \dots) \quad (4)$$

for $z \rightarrow \pm\infty$. Eq. (4) suggests that $\rho(\Delta v, \Delta x)$ approaches

power-law form as $\Delta x \rightarrow 0^+$

$$\rho(\Delta v, \Delta x) = |\Delta x|^{\mu-2} Z_\mu(\Delta v/\Delta x) \sim A_\mu^\pm (\pm \Delta v)^{\mu-2} \quad (5)$$

for positive and negative values of Δv , respectively. Now we use that interchanging the particles in a particle pair ($\Delta x \rightarrow -\Delta x$ and $\Delta v \rightarrow -\Delta v$) cannot alter the distribution. In particular this must be true at $\Delta x = 0$ which requires $A_\mu^+ = A_\mu^-$. The allowed values of μ can be computed by evaluating (3) and imposing that $A_\mu^+ = A_\mu^-$. To simplify the problem we only consider the smallest positive allowed value of μ (referred to as μ_c , and shown in Fig. 2c). This yields an accurate description (except in the very far tails) of the distribution $\rho(\Delta v, \Delta x)$ for small values of Δx as Fig. 2a shows.

Three regions can be distinguished in $\rho(\Delta v, \Delta x)$. First, the body of the distribution (labeled 1 in Fig. 2a) corresponds to small values of z ($|z| \ll 1$ that is $|\Delta v| \ll |\Delta x|$). Typically, the separation remains constant in this region, $\Delta x \approx \Delta x_0$, and Δv obeys $d\Delta v/dt \approx -\Delta v + \Delta u$. In the limit $Ku \rightarrow 0$, $St \rightarrow \infty$ so that $\epsilon^2 = Ku^2 St$ remains constant, this is an Ornstein-Uhlenbeck process. It gives rise to a broad Gaussian distribution of relative velocities, approximately independent of Δv . This smooth contribution is not universal.

Second, the regime $|z| \gg 1$ (that is $|\Delta v| \gg |\Delta x|$, labeled 2 in Fig. 2a) corresponds to particles approaching each other on different caustic branches. This singular contribution, due to caustics, yields large values of $|\Delta v|$ as $|\Delta x| \rightarrow 0$. In this case, the deterministic part of Eq. (2) dominates, giving rise to linear trajectories $\Delta v = \Delta v_0 + \Delta x_0 - \Delta x$. When $|\Delta v| \gg |\Delta x|$, we have $\Delta v \approx \Delta v_0$. In this limit, the distribution $\rho(\Delta v, \Delta x)$ becomes approximately independent of Δx . As shown above, it exhibits the power-law form (5). The power-law is clearly visible in Fig. 2a. This singular caustic contribution to $\rho(\Delta v, \Delta x)$ gives rise to large moments of Δv , and thus to a large collision rate. We emphasise that the dynamics in this regime is universal, that is independent of the particular statistics of the velocity field. Fig. 2b shows a typical phase-space trajectory passing through the regimes 1 and 2 and contributing to the smooth and singular parts of the distribution $\rho(\Delta v, \Delta x)$.

Third, as Fig. 2a shows, the power law is cut off at very large velocities, ensuring that the moments of Δv do not diverge. This region (labeled 3 in Fig. 2a) is not described by keeping the smallest positive value of μ_c only. The cut off, Δv_c , is simply due to the fact that the relative velocities Δv_0 at large separations are of order $\mathcal{D}_0^{1/2}$. This suggests that $\Delta v_c \sim \mathcal{D}_0^{1/2}$. The form of $\rho(\Delta v, \Delta x)$ at $|\Delta v| \gg \Delta v_c$ depends upon the details of the model. In turbulent flows at large Stokes and Reynolds numbers, Δv_c is determined by the ‘variable-range projection principle’ suggested in [12]. The cut off Δv_c corresponds to a cut off $\Delta v_c/\Delta x$ in the distribution of z . Thus $\rho(\Delta x, z)$ does not strictly factorise, except in the limit $\Delta x \rightarrow 0$. The probability distribution of z for large values of $|z|$ is obtained by integrating the joint distribution over Δx to $\pm \Delta v_c/|z|$. As a consequence of

the power law (4), the tails of $Z_0(z)$ found in [8] are recovered, namely $Z_0(z) \sim |z|^{-2}$. This shows once more that the distribution of the finite differences Δv and Δx is governed by the same principles as the fluctuations of $\partial v/\partial x$.

What do these results imply for the moments $m_p(\Delta x) = \int d\Delta v |\Delta v|^p \rho(\Delta v, \Delta x)$ at small Δx ? We find:

$$m_p(\Delta x) = |\Delta x|^{p+\mu_c-1} \int dz |z|^p Z_{\mu_c}(z) \sim |\Delta x|^{p+\mu_c-1} (b_p + c_p |\Delta x|^{-p-\mu_c+1}). \quad (6)$$

Here the coefficient b_p results from the contribution of the body of $Z_{\mu_c}(z)$ to the z -integral in (6). The second term is the caustic contribution, it results from integrating the power-law tails of $Z_{\mu_c}(z)$ up to the cut-off $\Delta v_c/\Delta x$, keeping only the leading-order behaviour in (4). This form of $m_p(\Delta x)$ was deduced from results of numerical simulations for the distribution of relative velocities and separations in a one-dimensional Kraichnan model [15], and it was suggested that the exponent μ_c equals the spatial correlation dimension. This is correct for $\mu_c < 1$, as the following argument shows. The spatial correlation dimension d_2 describes the power-law scaling for $p = 0$, namely $m_0(\Delta x) \sim |\Delta x|^{d_2-1}$ as $\Delta x \rightarrow 0$. From Eq. (6) we deduce $m_0(\Delta x) \sim |\Delta x|^{\min(\mu_c, 1)-1}$, implying that $\mu_c = d_2$ for $0 \leq \mu_c < 1$. But for $\mu_c > 1$ this is not the case. A change of variables $\Delta x \rightarrow \Delta w = \sqrt{\Delta x^2 + \Delta v^2}$ in (5) allows us to compute the distribution of ‘phase-space separations’ Δw . We find that it exhibits a power-law of the form Δw^{μ_c-1} as $\Delta w \rightarrow 0$. This proves that μ_c is in fact equal to the phase-space correlation dimension D_2 (which ranges between 0 and 2). It also demonstrates that $D_2 = d_2$ for $0 \leq D_2 \leq 1$ (consistent with results of numerical simulations of particles in two-dimensional random flows at finite Kubo numbers [16]). We computed spatial and phase-space correlation dimensions from numerical simulations of the equation of motion (1). The results are shown in Fig. 2c and are in good agreement with the analytical theory. Caustic contributions dominate for $p > 1 - \mu_c$, the smooth contribution is thus irrelevant for $p = 1$.

For two- and three-dimensional incompressible flows, the distribution of relative velocities $\Delta \mathbf{v}$ at spatial separation R has a form similar to (5): as $R \rightarrow 0$, the d -dimensional analogue of equation (2) is solved by the ansatz $g_\mu(R) Z_\mu(\mathbf{z})$ where $\mathbf{z} = \Delta \mathbf{v}/R$. We find that Z_μ exhibits tails of the form $Z_\mu(\mathbf{z}) \sim |\mathbf{z}|^{\mu-2d}$. Comparison with the expected form of the distribution of (phase-)space separations demonstrates that μ_c equals the phase-space correlation dimension. We find:

$$\rho(\Delta \mathbf{v}, R) \sim R^{d-1} |\Delta \mathbf{v}|^{D_2-2d}. \quad (7)$$

Now consider the distribution of relative radial velocities $V_R \equiv \Delta \mathbf{v} \cdot \hat{R}$ (\hat{R} is the radial unit vector). For $|V_R| \gg R$ this distribution is found by integration of (7). Assuming $D_2 < d+1$, we find for small values of R (and $|V_R| \gg R$)

$$\rho(V_R, R) \sim R^{d-1} |V_R|^{D_2-d-1}. \quad (8)$$

Eq. (8) yields the following expression for the moments of $m_p(R)$ of V_R :

$$m_p(R) = b_p(\text{St})R^{p+D_2-1} + c_p(\text{St})R^{d-1}, \quad (9)$$

of the same form as (6). But note that here the singular caustic term acquires a geometrical factor R^{d-1} . Setting $p = 0$ in (9) shows that $d_2 = \min\{D_2, d\}$ in analogy with the one-dimensional case. For $p = 1$, Eq. (9) is equivalent to the ansatz proposed in [3], apart from the difference that in Eq. (7) of [3], the asymptotic behaviour $D_2(\text{St}) \rightarrow d$ as $\text{St} \rightarrow 0$ was used for the first term in (9).

Fig. 2d shows numerical results for a linearised Langevin model for inertial particles in a two-dimensional incompressible Gaussian random flow. Shown is the distribution $\rho(V_R, R)$. As in the one-dimensional case, the power-laws in (7) and (8) represent the universal contribution of caustics to the distribution of relative velocities at small separations. This demonstrates how caustics and fractal clustering determine the distribution of relative velocities at small separations. Our numerical results imply that the prefactor $c_p(\text{St})$ of the singular caustic term in (9) is of activated form $c_p(\text{St}) \sim e^{-A_p/\epsilon^2}$ for small ϵ (not shown).

In the remainder we show how our theory explains the results of direct numerical simulations of particles suspended in turbulent flows [9]. In [9], particle-velocity structure functions $S_p(R) = m_p(R)/m_0(R)$ were computed numerically and analysed in terms of the scaling ansatz $S_p(R) \sim R^{\xi_p}$. Fig. 3 in [9] shows the scaling exponent ξ_1 as a function of St . Our theory, Eq. (9), predicts that $\xi_1 \approx 1$ for small values of St , and $\xi_1 = d - d_2$ for large values of St . These predictions are in quantitative agreement with the results presented in Fig. 3 in [9]. Our theory also explains the distribution of $z_R = v_R/R$ shown in Fig. 5 in [17]. Eq. (8) yields $P(z_R|R) \sim |z_R|^{D_2-d-1}$ which explains the power law in the inset of Fig. 5 in [9]. The far tails in this figure are expected to be determined by the variable-range projection principle [12] yielding $P(z_R|R) \sim \exp(-C|z_R R|^{4/3})$. As a next step it is necessary to determine whether the above expression describes the functional form of the far tails of this distribution in [17].

Financial support by Vetenskapsrådet, by the Göran Gustafsson Stiftelse, and by the platform for ‘Nanoparticles in interactive environments’ is gratefully acknowledged.

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